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Exercises for 'Functional Analysis 2' [MATH-404]

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Ex 3.1 (On sequences and boundedness in LCTVS)
Let X be a LCTVS with its topology being induced by a family of seminorms $(p_i)_{i\in I}$.

a) Show that a sequence $(x_n)_{n\in\mathbb{N}} \subset X$ converges to $x \in X$ if and only if $p_i(x_n - x) \to 0$ for all $i \in I$.

A set $E \subset X$ is called bounded if for every neighborhood U of 0 there exists $s > 0$ such that $E \subset sU$.

b) Show that a set $E \subset X$ is bounded if and only if $p_i(E)$ is a bounded subset of $\mathbb{R}$ for every $i \in I$.

A sequence $(x_n)_{n\in N} \subset X$ is called a **Cauchy sequence** if for every neighborhood U of 0 there exists $N \in \mathbb{N}$ such that $x_m - x_n \in U$ for all $n, m \geq N$.

c) Show that Cauchy sequences are bounded.
Hint: If U is a neighborhood of the origin, due to the continuity of the addition and Ex. 1.3 there exists an absorbing, balanced neighborhood V of 0 such that $V + V \subset U$.

Solution 3.1 : a) Assume that $x_n \to x$. Since X is a topological vector space, the translation-invariance of neighborhoods implies that $x_n - x \to 0$. Given $i \in I$ we define the open set $B_{\varepsilon,i}(0) = \{y \in X : p_i(y) < \varepsilon\}$ which contains 0. Hence by the definition of convergence we deduce that there exists $N = N(i, \varepsilon)$ such that for all $n \geq N$ it holds that $x_n - x \in B_{\varepsilon,i}(0)$, or equivalently, $p_i(x_n - x) < \varepsilon$. This implies that $p_i(x_n - x) \to 0$.

Now we suppose that $p_i(x_n-x) \to 0$ for all $i \in I$. In particular, for all finite sets $I_0 \subset I$ and $\varepsilon > 0$ there exists $N = N(I_0, \varepsilon)$ such that for all $n \geq N$ we have $\max_{i\in I_0} p_i(x_n - x) < \varepsilon$. In particular, for all $n \geq N$ it holds that $x_n - x \in B_{\varepsilon,I_0}(0)$. Since the latter balls define a neighborhood basis of the origin, it follows that $x_n - x \to 0$ and again by translational invariance $x_n \to x$.

b) Assume that $E \subset X$ is bounded. Since the ball $B_{1,i}(0)$ is a neighborhood of 0, it follows that there exists $s > 0$ such that $E \subset sB_{1,i}(0) = B_{s,i}(0)$. Hence $p_i(x) < s$ for all $x \in E$.

Next suppose that for every $i \in I$ there exists $k_i > 0$ such that $|p_i(x)| < k_i$ for all $x \in E$. Let $U \subset X$ be a neighborhood of 0. Then there exists $\varepsilon > 0$ and $I_0 \subset I$ finite such that $B_{\varepsilon,I_0}(0) \subset U$. Setting $s = \max_{i\in I_0} k_i/\varepsilon > 0$, it follows that for all $x \in E$ and $i \in I_0$,

$$p_i(x) < k_i \leq s\varepsilon$$

so that $x \in B_{s\varepsilon,I_0}(0) = sB_{\varepsilon,I_0}(0) \subset sU$ for all $x \in E$. Hence E is bounded.

c) Let $U \subset X$ be a neighborhood of the origin. We use the hint to find a balanced, absorbing, neighborhood V of 0 such that $V + V \subset U$. Since $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, there exists

$N \in \mathbb{N}$ such that for all $n \geq N$ it holds that $x_n - x_N \in V$. Since V is absorbing, we find $s_N > 0$ such that $x_N \in s_N V$. Hence, for $n \geq N$ and $t > \max\{1, s_N\}$ we find that

$$x_n = (x_n - x_N) + x_N \in V + s_N V \subset \max\{1, s_N\}V + \max\{1, s_N\}V \subset tV + tV \subset tU,$$

where we used the fact that the set λV is balanced for every $\lambda \in \mathbb{K}$, so that $sV \subset tV$ for all $s < t$. Finally, for $x_1, \ldots, x_{N-1}$, the continuity of the scalar multiplication implies that there exists $R > 0$ such that for all $j = 1, \ldots, N-1$ and $t \geq R$ it holds that $x_j \in tU$. Hence for all $t \geq \max\{1, s_N, R\}$ we have

$$\{x_n : n \in \mathbb{N}\} \subset tU$$

Note that we can also prove this without the hint using part (b). Indeed, since $(x_n)_{n\in\mathbb{N}}$ is Cauchy, for any $i \in I$, $\varepsilon > 0$, there exists $N > 0$ such that for all $n, m \geq N$ we have $x_n - x_m \in B_{\varepsilon,\{i\}}(0)$, i.e. $p_i(x_n - x_m) < \varepsilon$. Then since for any semi-norm p we have

$$|p(x) - p(y)| \leq p(x - y)$$

we deduce that for each $i \in I$ the sequence $(p_i(x_n))_{n\in\mathbb{N}}$ is Cauchy in $\mathbb{R}$, and hence bounded. Hence $\{x_n : n \in \mathbb{N}\}$ is bounded by (b). □

Ex 3.2 (On bounded and compact sets in TVS)
Let X be a TVS. Show that :

a) Every compact set $K \subset X$ is bounded.

Hint: Every neighborhood of 0 contains an absorbing and balanced neighborhood of 0, see Exercise 1.3 b)-c).

b) A set $E \subset X$ is bounded if and only if every countable subset of E is bounded.

c) If $E, F \subset X$ are bounded, so is $E + F$. If $E, F \subset X$ are compact, so is $E + F$.

d)* If $K \subset X$ is compact and $C \subset X$ is closed then $K + C$ is closed. Give an example of two closed sets in a TVS such that their sum is not closed.

Hint: Prove first that if K is contained in an open set U then there is an open neighborhood of 0 such that $K + V \subset U$.

Solution 3.2 :
a) Let $K \subset X$ be compact and let U be a neighborhood of 0. We need to show that there is $s > 0$ such that $K \subset sU$. To this end, choose an absorbing and balanced neighborhood V of 0 such that $V \subset U$ (see Exercise 1.3, items b)-c)). Because V is absorbing, we have

$$K \subset \bigcup_{n=1}^{\infty} nV = X.$$

Owing to the compactness of K, there is $N \in \mathbb{N}$ such that

$$K \subset \bigcup_{n=1}^{N} nV = N\Big(\bigcup_{n=1}^{N} \frac{n}{N}V\Big) \subset N\Big(\bigcup_{n=1}^{N} V\Big) = NV,$$

where the second inclusion holds since V is balanced. Because V is a subset of U, we get $K \subset sU$ with $s = N$.

b) Suppose for contradiction that E is not bounded but every countable subset is. Then there exists a neighborhood U of 0 such that $E \not\subset tU$ for any $t > 0$. Let $V \subset U$ be a balanced neighbourhood of the origin (which exists by Exercise 1.3). Then for any $n \in \mathbb{N}$ there exists

$x_n \in E \backslash nV$. The set $\{x_n : n \in \mathbb{N}\} \subset E$ is countable so bounded, so there exists $r > 0$ such that $\{x_n : n \in \mathbb{N}\} \subset rV$. Moreover, since V is balanced, for any $r' > r$ we have

$$rV = r'\Big(\frac{r}{r'}V\Big) \subset r'V.$$

In particular, for any $n > r$, we have $x_n \in rV \subset nV$ which is a contradiction.
Conversely, if E is bounded, then for any countable subset $\{x_n : n \in \mathbb{N}\} \subset E$ and any neighbourhood U of the origin there exists $s > 0$ such that $\{x_n : n \in \mathbb{N}\} \subset E \subset sU$, so $\{x_n : n \in \mathbb{N}\}$ is bounded.

c) Assume that $E, F \subset X$ are bounded. Let U be a neighborhood of 0 and take also a balanced neighborhood V of 0 such that $V + V \subset U$. That we can alway find such V for a given U follows from the continuity of the addition map (see the hint to the previous exercise).

By assumption, we can find $s_E, s_F > 0$ such that $E \subset s_E V$ and $F \subset s_F V$. Put $s := \max\{s_E, s_F\}$. Then, using balancedness of V

$$E + F \subset s_E V + s_F V = s\Big(\frac{s_E}{s}V\Big) + s\Big(\frac{s_F}{s}V\Big) \subset sV + sV = s(V + V) \subset sU.$$

Now, assume that $E, F \subset X$ are both compact. If we denote by $\mathcal{A}\colon X \times X \to X$ the addition map in X (i.e., $\mathcal{A}(x, y) = x + y$), we have

$$E + F = \mathcal{A}(E \times F).$$

By assumption, the set $E \times F$ is compact in the product topology of $X \times X$ (please verify that as a simple exercise in topology). Since $\mathcal{A}$ is continuous and $E + F$ is the image through $\mathcal{A}$ of a compact set, $E + F$ is compact too.

d) First we show that, if $K \subset X$ is compact and contained in an open set U, then there exists an open neighborhood of 0 such that $K + V \subset U$. (sidenote : the set $K + V$ is called the V**-neighborhood** of K because it is open and obviously $K \subset K + V$.)

Let K and U be as above. Since U is open and $K \subset U$, for each $x \in K$ there exists an open neighborhood V_x of 0 such that $x + V_x \subset U$. Moreover, by continuity of addition, there exists a neighborhood W_x of 0 such that $W_x + W_x \subset V_x$. Since K is compact, a finite number $x_1 + W_{x_1}, \ldots, x_n + W_{x_n}$ of these sets covers K. Let $V = \bigcap_{m=1}^{n} W_{x_m}$.

To show that $K + V \subset U$, consider any $x \in K$. There must be some integer m for which $x \in x_m + W_{x_m}$. Hence

$$x + V \subset x_m + W_{x_m} + V \subset x_m + W_{x_m} + W_{x_m} \subset x_m + V_{x_m} \subset U.$$

Therefore $K + V \subset U$.

Now we move to the main part of the exercise. Let K be compact, C closed and let $x \in \overline{K + C}$. Then, for all neighborhoods V of 0,

$$(x + V) \cap (K + C) \neq \emptyset \quad \iff \quad (-K + x + V) \cap C \neq \emptyset. \tag{$\star$}$$

To show that $x \in K + C$, first observe that if $(-K + x) \cap C = \emptyset$ then the compact set $-K + x$ is contained in an open set $X \setminus C$. By the first part, there exists an open neighborhood V of 0 such that $(-K + x + V) \subset X \setminus C$, which contradicts $(\star)$.

Finally, consider $X = \mathbb{R}$ with the usual Euclidean topology. The sets $A = \{n \in \mathbb{N} : \ n \geq 2\}$ and $B = \{-n - 1/n : \ n \geq 2\}$ are both closed, but the set $A + B$ contains $\{-1/n : \ n \in \mathbb{N}\}$ which has 0 as an accumulation point, so $A + B$ is not closed.

Ex 3.3 (On the continuity of seminorms on LCTVS)
Let X be a locally convex topological vector space with seminorms $(p_i)_{i \in I}$ generating the topology. Consider another seminorm $q : X \to [0, +\infty)$. Show that the following properties are equivalent :

i) q is continuous.

ii) There exist $c > 0$ and $I_0 \subset I$ finite such that

$$q(x) \leq c \sum_{i \in I_0} p_i(x).$$

Solution 3.3 : i) $\implies$ ii) : Assume that q is continuous. Then it is particularly continuous in 0, so that there exists $\delta > 0$ and $I_0 \subset I$ finite such that $q(x) < 1$ for all $x \in B_{\delta, I_0}(0)$. In particular, for any $x \in X$ we distinguish the following two cases :

1) $P_{I_0}(x) := \sum_{i \in I_0} p_i(x) = 0$: in this case, we have for all $R > 0$ that $P_{I_0}(Rx) = RP_{I_0}(x) = 0$, so that in particular $Rx \in B_{\delta, I_0}(0)$ and therefore $Rq(x) = q(Rx) < 1$ for all $R > 0$. This implies $q(x) = 0$, so that $q(x) \leq cP_{I_0}(x)$ for all $c > 0$;

2) $P_{I_0}(x) > 0$: Then $y := \delta x / (2P_{I_0}(x))$ satisfies $y \in B_{\delta, I_0}(0)$, so that $q(y) < 1$, which yields

$$1 > q(y) = \frac{\delta}{2P_{I_0}(x)} q(x),$$

which implies the claim with $c = \frac{2}{\delta}$.

ii) $\implies$ i) : We first show that q is continuous in 0. Denote by k the cardinality of I_0. Then for any $\varepsilon > 0$ it holds that

$$B_{\frac{\varepsilon}{ck}, I_0}(0) \subset q^{-1}((-\varepsilon, \varepsilon)).$$

Hence by definition q is continuous in 0. Next we prove that q is continuous in an arbitrary point $x \in X$. To this end, note that when V is a neighborhood of the origin such that $q(V) \subset (-\varepsilon, \varepsilon)$, then $x + V$ is a neighborhood of x such that for any $y \in x + V$ it holds that

$$|q(y) - q(x)| \leq q(y - x) < \varepsilon,$$

so that q is continuous in x. Hence q is continuous.

Ex 3.4 (On test functions on compact intervals*)
Let $[a, b]$, where $a < b$, be a compact interval in $\mathbb{R}$. Consider the vector space

$$\mathcal{D}_{[a,b]} = \{f \in C^\infty(\mathbb{R}) : \ \text{supp}(f) \subseteq [a, b]\},$$

where $C^\infty(\mathbb{R})$ is the space of all smooth functions on $\mathbb{R}$ and $\text{supp}(f)$ is the **support** of f (namely, the complement of the largest open set on which f vanishes).

a) Show that the function

$$\phi(x) = \begin{cases} e^{-1/t} & \text{for } t > 0 \\ 0 & \text{for } t \leq 0 \end{cases}$$

belongs to $C^\infty(\mathbb{R})$. As a consequence, notice that $f(x) = \phi(x - a)\phi(b - x)$ is in $\mathcal{D}_{[a,b]}$.

b) Consider the family of seminorms

$$p_n(f) = \max\left\{\left|f^{(k)}(x)\right| : \ x \in \mathbb{R}, \ k \le n\right\}, \quad f \in \mathcal{D}_{[a,b]}, \ n = 0, 1, \dots,$$

where $f^{(k)}$ stands for the k-th derivative of f. Show that this family introduces a locally convex topology on $\mathcal{D}_{[a,b]}$. What is the neighborhood basis of 0 in this topology. What does it mean that a sequence $\{f_n\} \subset \mathcal{D}_{[a,b]}$ converges to $f \in \mathcal{D}_{[a,b]}$? Is this topology metrizable/normable?

c) Let $E \subset \mathcal{D}_{[a,b]}$ be closed and bounded in the topology from b). Show that for every $k = 0, 1, \dots$, the set $\{f^{(k)} : \ f \in E\}$ is a precompact subset of $C([a,b])$ (the space of continuous functions on $[a,b]$ with max norm). Using this fact, demonstrate that E is compact in $\mathcal{D}_{[a,b]}$.

Hint: For the first part use Arzelà–Ascoli theorem, for the second Cantor's diagonal argument. *(Let us know if you are not familiar with these tools)*